An Introduction to Loose Legendrians in High Dimensions

Kylerec seminar

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Symplectic & contact manifolds

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An isotropic submanifold Λ in (M, ω) (resp. (Y, α)) is a submanifold such that $\omega|_\Lambda = 0$ (resp. $\alpha|_\Lambda = 0$). When $\dim \Lambda = n$, it is called a Lagrangian (resp. Legendrian) submanifold.

Darboux & Weinstein theorems

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For any point in a symplectic (resp. contact) manifold, there is a neighbourhood that is symplectomorphic (resp. contactomorphic) to an open subset in $\mathbb{R}^{2n}_{std}$ (resp. $\mathbb{R}^{2n+1}_{std}$).

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There are no 'local invariants' in symplectic/contact topology. They are more 'flexible' (more like topology).

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- **Answer:** Yes. There are some global flexibility results called *h*-principles.
- **Question again:** What is an *h*-principle?
- **Answer:**Roughly speaking, *h*-principle enables one to reduce a symplectic/contact topology problem to an algebraic topology problem.

An example in smooth topology

Proposition

For any $n \in \mathbb{Z}$, *there is an immersion* $f : S^1 \to \mathbb{R}^2$ *with* $\mathrm{rot}(Df) = n$ *(existence).*
Any immersions $f_0, f_1 : S^1 \to \mathbb{R}^2$ *such that* $\mathrm{rot}(Df_0) = \mathrm{rot}(Df_1)$ *are homotopic through immersions (uniqueness).*

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1. decouple the map f and its derivative df, extract the information separately and get necessary conditions;
2. show that this is enough.

What is an h-principle?

Suppose we want to find a map $f : M \to N$ such that

$$P(f, Df, D^2 f, ..., D^k f) = 0 \text{ or } P(f, Df, ..., D^k f) \neq 0.$$

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If step 2 works, then we say that h-principle holds in this case.

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Definition

An formal isotropic embedding $\Lambda \to M$ (resp. $\Lambda \to Y$) is a smooth embedding $f : \Lambda \to M$ and an isotropic bundle map $F : T\Lambda \to f^*TM$ (resp. $F : T\Lambda \to f^*\xi \hookrightarrow f^*TY$) that is homotopic to Df.

h-principle for isotropic submanifolds

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Lagrangian/Legendrian immersions into symplectic/contact manifolds satisfy all h-principles.

Questions: What about Lagrangian/Legendrian embeddings?

h-principle for loose Legendrians

Not all Legendrian submanifolds are flexible, and there are invariants defined by pseudo-holomorphic curves (Legendrian contact homology). However, in dimension at least 5, there is a class of Legendrians that are flexible:

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Theorem (Murphy 2012)

When the dimension of the contact manifold is at least 5*, there is a class of Legendrian submanifolds called loose Legendrians that satisfy all h-principles.*

How do we characterize Legendrians?

Consider the standard contact manifold $(\mathbb{R}^{2n+1}, \alpha_{\text{std}} = dz - \sum_{i=1}^{n} y_i dx_i)$.
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The front projection in the contact manifold $\mathbb{R}^{2n+1}_{\text{std}}$ is

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There cannot be vertical tangencies, but there can be singularities like cusps $(x^2, 3x, 2x^3)$ in $\mathbb{R}^3$.

Definition of loose Legendrians

Definition

Let $I^3 \subset \mathbb{R}^3$ be a cube of side length 1, $\Lambda_0 \subset I^3$ be a Legendrian curve whose front projection is a zig-zag and is equal to $\{(x,y,z)|y=z=0\}$ near ∂I^3. Let $\rho > 1$. $D_\rho = \{(q,p) \in \mathbb{R}^{2(n-1)} || p| \leq \rho, |q| \leq \rho\}$. $Z_\rho = \{(q,p)|p=0, |q| \leq \rho\}$. Then a standard loose chart is

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2. Why do we require $\rho > 1$ for the contact neighbourhood $I^3 \times D_\rho$?
3. Why doesn't it work in dimension 3?

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Fuchs-Tabachnikov (1997) showed that one can get Legendrian links in dimension 3 by adding zig-zags (called stablizations).

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Remark

However, this construction in dimension 3 changes the formal data, i.e. the genuine Legendrian on the right is not homotopic to the formal Legendrian on the left in the jet bundle.

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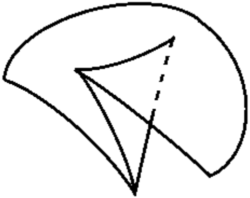

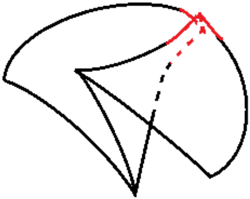

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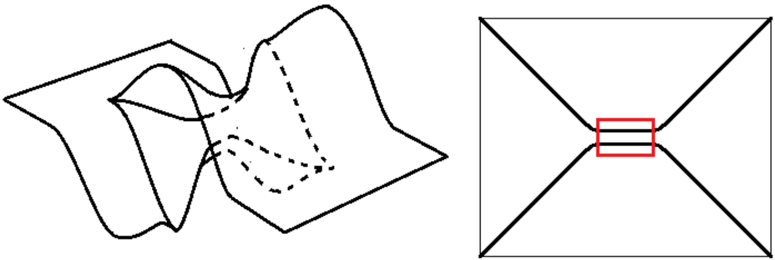

Figure: Shrinking the loose chart to get a long tube containing arbitrarily many loose charts. The red region contains many loose charts.

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- **Question:** If the Legendrian contact homology is 0, is the Legendrian necessarily loose?
- Murphy-Siegel and Lazarev-Sylvan essentially showed that for some contact manifolds (that are boundaries of certain Liouville manifolds), this is not true. However, the question is still open in many important cases, e.g. for Legendrians in $\mathbb{R}^{2n+1}_{\text{std}}$ or S^{2n+1}_{std}.

Applications to Liouville manifolds

- Consider the loose Legendrian sphere $\Lambda \subset S^{2n+1}_{\text{std}}$ (that is formally isotopic to the unknotted sphere). By attaching a handle $D^n \times D^n$ to Λ, we get a Liouville (Weinstein) manifold that is diffeomorphic to T^*S^n (by h-principle) but has zero wrapped Fukaya category.

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- Using Kirby calculus (for Weinstein manifolds), Casals-Murphy and Lazarev can simplify the handlebody decomposition of certain Liouville manifolds and get a number of cool results, e.g.

$$X_{1,b} = \left\{ xy^b + \sum_{i=1}^{n-1} z_i^2 = 1 \right\} \subset \mathbb{C}^{n+1},\ \ b \geq 2$$

are all symplectomorphic and have 0 wrapped Fukaya categories/symplectic cohomologies.

Displacement energy

- Given a Legendrian Λ, what is the C^0-norm of the Hamiltonian that is required to displace Λ, so that there are no Reeb chords between Λ and $\varphi_H^1(\Lambda)$?

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- Given a Legendrian Λ, what is the C^0-norm of the Hamiltonian that is required to displace Λ, so that there are no Reeb chords between Λ and $\varphi_H^1(\Lambda)$?
- Nakamura showed that, for loose Legendrians, the norm only depends on the loose chart. Dimitroglou Rizell and Sullivan's work implies that loose Legendrians are the easiest to be displaced.

Other applications

- Ekholm-Eliashberg-Murphy-Smith used loose Legendrians to create exact Lagrangian immersions with few double points.